

# A Description of the Subgraph Induced at a Labeling of a Graph by the Subset of Vertices with an Interval Spectrum.

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## Abstract

The sets of vertices and edges of an undirected, simple, finite, connected graph  $G$  are denoted by  $V(G)$  and  $E(G)$ , respectively. An arbitrary nonempty finite subset of consecutive integers is called an interval. An injective mapping  $\varphi : E(G) \rightarrow \{1, 2, \dots, |E(G)|\}$  is called a labeling of the graph  $G$ . If  $G$  is a graph,  $x$  is its arbitrary vertex, and  $\varphi$  is its arbitrary labeling, then the set  $S_G(x, \varphi) \equiv \{\varphi(e)/e \in E(G), e \text{ is incident with } x\}$  is called a spectrum of the vertex  $x$  of the graph  $G$  at its labeling  $\varphi$ . For any graph  $G$  and its arbitrary labeling  $\varphi$ , a structure of the subgraph of  $G$ , induced by the subset of vertices of  $G$  with an interval spectrum, is described.

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# 1 Introduction

We consider undirected, simple, finite and connected graphs, containing at least one edge. The terms and concepts which are not defined can be found in [1].

For a graph  $G$ , we denote by  $V(G)$  and  $E(G)$  the sets of its vertices and edges, respectively. The set of vertices of  $G$  adjacent to a vertex  $x \in V(G)$  is denoted by  $I_G(x)$ . The set of edges of  $G$  incident with a vertex  $x \in V(G)$  is denoted by  $J_G(x)$ .

If  $G$  is a graph, and  $x \in V(G)$ , then  $d_G(x)$  denotes the degree of the vertex  $x$  in the graph  $G$ . For any graph  $G$ ,  $\Delta(G)$  denotes the maximum degree of a vertex of  $G$ .

For any graph  $G$ , we define the subsets  $V'(G)$  and  $V''(G)$  of its vertices by the following way:  $V'(G) \equiv \{x \in V(G) / d_G(x) = 1\}$ ,  $V''(G) \equiv \{x \in V(G) / d_G(x) \geq 2\}$ .

The distance in a graph  $G$  between its vertices  $x$  and  $y$  is denoted by  $d_G(x, y)$ .

For any graph  $G$ , we denote by  $diam(G)$  its diameter. A vertex  $x \in V(G)$  is called a peripheral vertex of a graph  $G$  if there exists a vertex  $y \in V(G)$  satisfying the condition  $d_G(x, y) = diam(G)$ .

For an arbitrary nonempty finite subset  $A$  of the set  $\mathbb{Z}_+$ , we denote by  $l(A)$  and  $L(A)$ , respectively, the least and the greatest element of  $A$ .

An arbitrary nonempty finite subset  $A$  of the set  $\mathbb{Z}_+$  is called an interval if it satisfies the condition  $|A| = L(A) - l(A) + 1$ . An interval with the minimum element  $p$  and the maximum element  $q$  is denoted by  $[p, q]$ .

An injective mapping  $\varphi : E(G) \rightarrow \mathbb{N}$  is called a labeling of the graph  $G$ . For any graph  $G$ , we denote by  $\psi(G)$  the set of all labelings of the graph  $G$ .

If  $G$  is a graph,  $\varphi \in \psi(G)$ , and  $E_0 \subseteq E(G)$ , we set

$$\varphi[E_0] \equiv \bigcup_{e \in E_0} \{\varphi(e)\}.$$

If  $G$  is a graph,  $x \in V(G)$ , and  $\varphi \in \psi(G)$ , then the set  $\varphi[J_G(x)]$  is called a spectrum of the vertex  $x$  of the graph  $G$  at the labeling  $\varphi$ .

If  $G$  is a graph, and  $\varphi \in \psi(G)$ , then we set  $U(G, \varphi) \equiv \{x \in V(G) / \varphi[J_G(x)] \text{ is an interval}\}$ .

If  $G$  is a graph, and  $\varphi \in \psi(G)$ , then we denote by  $G^{(\varphi, int)}$  the subgraph of the graph  $G$  induced by the subset  $U(G, \varphi)$  of its vertices.

For any graph  $G$ , we set  $\lambda(G) \equiv \{\varphi \in \psi(G) / U(G, \varphi) \neq \emptyset\}$ . Clearly, for any graph  $G$ ,  $\lambda(G) \neq \emptyset$ .

If  $G$  is a graph,  $\varphi \in \lambda(G)$ ,  $(x_0, x_1) \in E(G)$ , then the simple path  $P = (x_0, (x_0, x_1), x_1)$  is called a trivial  $\varphi$ -gradient path of  $G$  iff the following two conditions hold:

1.  $\{x_0, x_1\} \subseteq U(G, \varphi)$ ,
2. at least one of the following two conditions holds:

- (a)  $\varphi((x_0, x_1)) = L(\varphi[J_{G(\varphi, \text{int})}(x_0)]) = l(\varphi[J_{G(\varphi, \text{int})}(x_1)])$ ,
- (b)  $\varphi((x_0, x_1)) = l(\varphi[J_{G(\varphi, \text{int})}(x_0)]) = L(\varphi[J_{G(\varphi, \text{int})}(x_1)])$ .

If  $G$  is a graph,  $\varphi \in \lambda(G)$ , then a simple path  $P = (x_0, (x_0, x_1), x_1, \dots, x_k, (x_k, x_{k+1}), x_{k+1})$  with  $k \in \mathbb{Z}_+$  is called a  $\varphi$ -gradient path of  $G$ , if either  $k = 0$  and  $P$  is a trivial  $\varphi$ -gradient path of  $G$ , or  $k \in \mathbb{N}$  and the following two conditions hold:

1.  $V(P) \subseteq U(G, \varphi)$ ,
2. exactly one of the following two conditions holds:
  - (a) for any  $i \in [0, k]$ ,  $\varphi((x_i, x_{i+1})) = L(\varphi[J_{G(\varphi, \text{int})}(x_i)]) = l(\varphi[J_{G(\varphi, \text{int})}(x_{i+1})])$ ,
  - (b) for any  $i \in [0, k]$ ,  $\varphi((x_i, x_{i+1})) = l(\varphi[J_{G(\varphi, \text{int})}(x_i)]) = L(\varphi[J_{G(\varphi, \text{int})}(x_{i+1})])$ .

If  $G$  is a graph,  $\varphi \in \lambda(G)$ , then the set of all  $\varphi$ -gradient paths of  $G$  is denoted by  $\xi(G, \varphi)$ .

If  $G$  is a graph,  $\varphi \in \lambda(G)$ , and  $P \in \xi(G, \varphi)$ , then  $P$  is called a maximal  $\varphi$ -gradient path of  $G$ , if there is no  $\tilde{P} \in \xi(G, \varphi)$  with  $V(P) \subset V(\tilde{P})$ .

If  $G$  is a graph,  $\varphi \in \lambda(G)$ , then the set of all maximal  $\varphi$ -gradient paths of  $G$  is denoted by  $\tau(G, \varphi)$ .

For arbitrary integers  $n$  and  $i$ , satisfying the inequalities  $n \geq 3$ ,  $2 \leq i \leq n-1$ , and for any sequence  $A_{n-2} \equiv (a_1, a_2, \dots, a_{n-2})$  of nonnegative integers, we define the sets  $V[i, A_{n-2}]$  and  $E[i, A_{n-2}]$  as follows:

$$V[i, A_{n-2}] \equiv \begin{cases} \{y_{i,1}, \dots, y_{i,a_{i-1}}\}, & \text{if } a_{i-1} > 0 \\ \emptyset, & \text{if } a_{i-1} = 0, \end{cases}$$

$$E[i, A_{n-2}] \equiv \begin{cases} \{(x_i, y_{i,j}), / 1 \leq j \leq a_{i-1}\}, & \text{if } a_{i-1} > 0 \\ \emptyset, & \text{if } a_{i-1} = 0. \end{cases}$$

For any integer  $n \geq 3$ , and for any sequence  $A_{n-2} \equiv (a_1, a_2, \dots, a_{n-2})$  of nonnegative integers, we define a graph  $T[A_{n-2}]$  as follows:

$$V(T[A_{n-2}]) \equiv \{x_1, \dots, x_n\} \cup \left( \bigcup_{i=2}^{n-1} V[i, A_{n-2}] \right),$$

$$E(T[A_{n-2}]) \equiv \{(x_i, x_{i+1}) / 1 \leq i \leq n-1\} \cup \left( \bigcup_{i=2}^{n-1} E[i, A_{n-2}] \right).$$

A graph  $G$  is called a galaxy, if either  $G \cong K_2$ , or there exist an integer  $n \geq 3$  and a sequence  $A_{n-2} \equiv (a_1, a_2, \dots, a_{n-2})$  of nonnegative integers, for which  $G \cong T[A_{n-2}]$ .

In the paper, for any graph  $G$  and arbitrary  $\varphi \in \lambda(G)$ , a structure of the subgraph  $G^{(\varphi, \text{int})}$  of the graph  $G$  is described. The main result was announced in [2].

## 2 Preliminary Notes

**Lemma 2.1** *If  $G$  is a graph,  $\varphi \in \lambda(G)$ ,  $\{x, y\} \subseteq U(G, \varphi)$ ,  $(x, y) \in E(G)$ , then  $|\varphi[J_G(x)] \cap \varphi[J_G(y)]| = 1$ .*

*Proof.* If  $\min\{d_G(x), d_G(y)\} = 1$ , the statement is evident. Now suppose that  $\min\{d_G(x), d_G(y)\} \geq 2$ . Since  $(x, y) \in E(G)$  we have  $|\varphi[J_G(x)] \cap \varphi[J_G(y)]| \geq 1$ . Let us assume that  $|\varphi[J_G(x)] \cap \varphi[J_G(y)]| \geq 2$ . It means that there exist  $e' \in J_G(x)$ ,  $e'' \in J_G(y)$ , which satisfy the conditions  $e' \neq (x, y)$ ,  $e'' \neq (x, y)$ ,  $e' \neq e''$ ,  $\varphi(e') = \varphi(e'')$ . It is incompatible with  $\varphi \in \lambda(G)$ .

*Lemma is proved.*

**Corollary 2.2** *Let  $G$  be a graph, and  $\varphi \in \lambda(G)$ . Suppose that vertices  $x_1, x_2, \dots, x_n$  ( $n \geq 2$ ) of the graph  $G$  satisfy the conditions*

1.  $\{x_1, \dots, x_n\} \subseteq U(G, \varphi) \cap V''(G)$ ,
2. for any  $i \in [1, n-1]$ ,  $(x_i, x_{i+1}) \in E(G)$ .

*Then exactly one of the following two statements is true:*

1. for any  $i \in [1, n-1]$ ,  $\varphi((x_i, x_{i+1})) = L(\varphi[J_G(x_i)]) = l(\varphi[J_G(x_{i+1})])$ ,
2. for any  $i \in [1, n-1]$ ,  $\varphi((x_i, x_{i+1})) = l(\varphi[J_G(x_i)]) = L(\varphi[J_G(x_{i+1})])$ .

**Corollary 2.3** *If  $G$  is a graph, and  $\varphi \in \lambda(G)$ , then  $G^{(\varphi, int)}$  is a forest.*

*Proof.* Assume the contrary: the graph  $G^{(\varphi, int)}$  contains a subgraph  $G_0$  which is isomorphic to a simple cycle. Clearly, there exists an edge  $e_0 = (x, y) \in E(G_0)$ , for which  $\varphi(e_0) = l(\varphi[E(G_0)])$ . From here, taking into account that  $\{x, y\} \subseteq U(G, \varphi) \cap V''(G)$ , we obtain  $|\varphi[J_G(x)] \cap \varphi[J_G(y)]| \geq 2$ . It contradicts lemma 2.1.

*Corollary is proved.*

**Lemma 2.4** *Let  $G$  be a graph, and  $\varphi \in \lambda(G)$ . Then, for an arbitrary vertex  $x \in U(G, \varphi)$ , the inequality  $|I_G(x) \cap U(G, \varphi) \cap V''(G)| \leq 2$  holds.*

*Proof.* Suppose that there exists a vertex  $z_0 \in U(G, \varphi)$  with  $|I_G(z_0) \cap U(G, \varphi) \cap V''(G)| \geq 3$ . Let us choose three different vertices  $y_1, y_2, y_3$  from the set  $I_G(z_0) \cap U(G, \varphi) \cap V''(G)$ .

Note that the vertices  $y_1, z_0, y_2$  of the graph  $G$  satisfy the conditions of corollary 2.2 (with  $y_1$  in the role of  $x_1$ ,  $z_0$  in the role of  $x_2$ ,  $y_2$  in the role of  $x_3$ , and with  $n = 3$ ).

Note also that the vertices  $y_1, z_0, y_3$  of the graph  $G$  satisfy the conditions of corollary 2.2 (with  $y_1$  in the role of  $x_1$ ,  $z_0$  in the role of  $x_2$ ,  $y_3$  in the role of  $x_3$ , and with  $n = 3$ ).

*Case 1.* For the vertices  $y_1, z_0, y_2$  of the graph  $G$ , the statement 1) of corollary 2.2 is true. It means that  $\varphi((y_1, z_0)) = L(\varphi[J_G(y_1)]) = l(\varphi[J_G(z_0)])$ ,  $\varphi((z_0, y_2)) = L(\varphi[J_G(z_0)]) = l(\varphi[J_G(y_2)])$ . It is not difficult to see that for the vertices  $y_1, z_0, y_3$  of the graph  $G$  also the statement 1) of corollary 2.2 is true. It means that  $\varphi((y_1, z_0)) = L(\varphi[J_G(y_1)]) = l(\varphi[J_G(z_0)])$ ,  $\varphi((z_0, y_3)) = L(\varphi[J_G(z_0)]) = l(\varphi[J_G(y_3)])$ . The equalities  $\varphi((z_0, y_2)) = L(\varphi[J_G(z_0)])$  and  $\varphi((z_0, y_3)) = L(\varphi[J_G(z_0)])$  are incompatible.

*Case 2.* For the vertices  $y_1, z_0, y_2$  of the graph  $G$ , the statement 2) of corollary 2.2 is true.

The proof is similar as in case 1.

*Lemma is proved.*

**Lemma 2.5** *Let  $G$  be a graph,  $\varphi \in \lambda(G)$ ,  $\xi(G, \varphi) \neq \emptyset$ ,  $P \in \xi(G, \varphi)$ . Then there exists a unique  $\tilde{P} \in \tau(G, \varphi)$  satisfying the condition  $V(P) \subseteq V(\tilde{P})$ .*

*Proof is evident.*

**Corollary 2.6** *Let  $G$  be a graph,  $\varphi \in \lambda(G)$ ,  $\{x, y\} \subseteq V''(G^{(\varphi, int)})$ ,  $(x, y) \in E(G)$ . Then there exists a unique  $\tilde{P} \in \tau(G, \varphi)$  satisfying the condition  $\{x, y\} \subseteq V(\tilde{P})$ .*

### 3 Main Result

**Theorem 3.1** [2] *For any graph  $G$  and arbitrary  $\varphi \in \lambda(G)$ ,  $G^{(\varphi, int)}$  is a forest, each connected component  $H$  of which satisfies one of the following two conditions: 1)  $H \cong K_1$ , and the only vertex of the graph  $H$  may or may not belong to the set  $V'(G)$ , 2)  $H$  is a galaxy satisfying one of the following three conditions: a)  $V'(H) \subseteq V'(G)$ , b) exactly one vertex of the set  $V'(H)$ , which is a peripheral vertex of  $H$ , doesn't belong to the set  $V'(G)$ , c) exactly two vertices of the set  $V'(H)$ , with  $diam(H)$  as the distance between them, don't belong to the set  $V'(G)$ .*

*Proof.* Choose an arbitrary  $\varphi \in \lambda(G)$ . Let us consider an arbitrary connected component  $H$  of the graph  $G^{(\varphi, int)}$ . By corollary 2.3,  $H$  is a tree.

*Case 1.*  $|V(H)| = 1$ . In this case there is nothing to prove.

*Case 2.*  $|V(H)| = 2$ . Clearly,  $H \cong K_2$ , and the proposition is evident.

*Case 3.*  $|V(H)| \geq 3$ . Clearly,  $|V''(H)| \geq 1$ .

*Case 3.1.*  $|V''(H)| = 1$ . In this case  $H \cong K_{\Delta(H), 1}$ ,  $|V'(H)| = \Delta(H) = |V(H)| - 1 \geq 2$ ,  $diam(H) = 2$ . Clearly,  $H$  is a galaxy, and all vertices of  $V'(H)$  are peripheral vertices of the graph  $H$ . Without loss of generality we can assume that  $V''(H) = \{u_0\}$ , and, moreover, that the vertices  $u' \in V'(H)$  and  $u'' \in V'(H)$  satisfy the conditions  $l(\varphi[E(H)]) = \varphi((u_0, u'))$ ,  $L(\varphi[E(H)]) = \varphi((u_0, u''))$ .

Now consider an arbitrary vertex  $z \in V'(H)$ , which satisfies the condition  $l(\varphi[E(H)]) < \varphi((u_0, z)) < L(\varphi[E(H)])$ .

Let us show that  $z \in V'(G)$ . Assume the contrary:  $z \in V''(G)$ . From here we obtain the inequality  $|\varphi[J_G(u_0)] \cap \varphi[J_G(z)]| \geq 2$ , which contradicts lemma 2.1.

Consequently,  $V'(H) \cap V''(G) \subseteq \{u', u''\}$ , and, therefore,  $0 \leq |V'(H) \cap V''(G)| \leq 2$ . It completes the proof of case 3.1.

*Case 3.2.*  $|V''(H)| \geq 2$ . Clearly, in this case there exist vertices  $x \in V''(H)$  and  $y \in V''(H)$  satisfying the condition  $(x, y) \in E(H)$ . By corollary 2.6, there exists a unique  $P_0 \in \tau(G, \varphi)$  with  $\{x, y\} \subseteq V(P_0)$ . Suppose that  $w'$  and  $w''$  are endpoints of  $P_0$ . It is not difficult to see that  $d_{P_0}(w', w'') \geq 3$  and  $|I_G(w') \cap U(G, \varphi)| = |I_G(w'') \cap U(G, \varphi)| = 1$ .

Let us show that

$$\left( \bigcup_{x \in V''(P_0)} I_H(x) \right) \setminus V(P_0) \subseteq V'(G).$$

If  $(\bigcup_{x \in V''(P_0)} I_H(x)) \setminus V(P_0) = \emptyset$ , the required relation is evident. Now assume that  $(\bigcup_{x \in V''(P_0)} I_H(x)) \setminus V(P_0) \neq \emptyset$ .

Choose an arbitrary vertex  $z \in (\bigcup_{x \in V''(P_0)} I_H(x)) \setminus V(P_0)$ . Let us show that  $z \in V'(G)$ . Assume the contrary:  $z \in V''(G)$ .

Consider the vertex  $z_0 \in V''(P_0)$  which is adjacent to  $z$ . From the properties of  $P_0$  it follows that  $l(\varphi[J_H(z_0)]) < \varphi((z_0, z)) < L(\varphi[J_H(z_0)])$ . Since  $\{z_0, z\} \subseteq U(G, \varphi)$  we obtain that  $|\varphi[J_G(z_0)] \cap \varphi[J_G(z)]| \geq 2$ . It contradicts lemma 2.1.

Thus, indeed,  $(\bigcup_{x \in V''(P_0)} I_H(x)) \setminus V(P_0) \subseteq V'(G)$ . It implies that  $V'(H) \cap V''(G) \subseteq \{w', w''\}$ ,  $0 \leq |V'(H) \cap V''(G)| \leq 2$ , and, that  $P_0$  is the unique path in the graph  $H$  between its vertices  $w'$  and  $w''$ . Now it is easy to see that  $\text{diam}(H) = \text{diam}(P_0) = d_{P_0}(w', w'') = d_H(w', w'')$ . It completes the proof of case 3.2.

*Theorem is proved.*

**Corollary 3.2** [2] *If  $G$  is a graph with  $V'(G) = \emptyset$ , and  $\varphi \in \lambda(G)$ , then an arbitrary connected component of the forest  $G^{(\varphi, \text{int})}$  is a simple path.*

**Corollary 3.3** [2] *A labeling, which provides every vertex of a graph  $G$  with an interval spectrum, exists iff  $G$  is a galaxy.*

**Corollary 3.4** *If  $n \in \mathbb{N}$ , and  $\varphi \in \lambda(K_n)$ , then the forest  $K_n^{(\varphi, \text{int})}$  is a tree which is isomorphic either to  $K_1$ , or to  $K_2$ .*

## References

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